(iii) $\|\alpha x\|_\infty = \sup\limits_{t\in J} |\alpha x(t)|$, $\alpha \in \mathbb{R}$.

$= |\alpha| \sup\limits_{t\in J} |x(t)|$

$= |\alpha| \|x\|_\infty \quad \forall\, \alpha \in \mathbb{R} \;\forall\, x \in C[a,b]$.

(iv) Let $x, y \in C[a,b]$

$\|x+y\|_\infty = \sup\limits_{t\in J} |(x+y)(t)|$

$= \sup\limits_{t\in J} |x(t)+y(t)| \quad \because x, y$ is cts

$\le \sup\limits_{t\in J} |x(t)| + \sup\limits_{t\in J} |y(t)|$

$= \|x\|_\infty + \|y\|_\infty$ (triangle inequality)

$\therefore (C[a,b], \|\cdot\|_\infty)$ is a Banach space. OR Complete normed space.

(*) $C[a,b] = \{ f \mid f : [a,b] \longrightarrow \mathbb{R}$ is cts $\}$.

$C^k[a,b] = \{ f \mid f : [a,b] \longrightarrow \mathbb{R}$ is k-differentiable $\Rightarrow f^k$ is cts on $[a,b]\}$

Eg: $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases} \notin C^1[a,b]$.

$\therefore f(x)$ is not differentiable.

$\because f'(0) = \lim\limits_{x\to 0} \dfrac{f(x)-f(0)}{x} = \lim\limits_{x\to 0} \dfrac{x^2 \sin\frac{1}{x}}{x} = \lim\limits_{x\to 0} x\sin\frac{1}{x}$

$= 0$